Optimization of Digitization-based Kater Pendulum for Precise Gravity Measurement

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Abstract

In the undergrad laboratory of a physics course, it has been known that the simple pendulum experiment can be used to calculate the gravitational acceleration (g). An accurate g contributes to greater accuracy among equipment that requires g values as an input. However for research or application purposes, especially in the field of metrology, access to accurate g measurement equipment is expensive. Indeed, because there is no access to highly accurate equipment in Thailand, researchers here must rely on equipment from abroad, such as China or European countries, which is very costly. Hence this research aims to develop a device for g measurement, namely the Kater pendulum. Not only can the Kater pendulum provide the required accuracy, it also is relatively easier to design with a limited budget. Accuracy and precision of measurement are the primary concerns for the system’s development. Our first prototype of the Kater pendulum used a laser sensor-based measurement of the oscillation period, which is the key parameter for g evaluation. It provided an approximate 0.021% deviation of g from the reference, which is relatively good compared to that of the typical equipment used in the classroom, about 0.8% or Kater pendulum using a typical standard stopwatch, about 0.25%. More optimization schemes to improve accuracy and precision of measurement are being developed. It is anticipated that the optimal version of this equipment will greatly benefit many physics researchers, particularly those in the field of metrology.

Keywords: Kater pendulum, reversible pendulum, pendulum
Optimization of digitization based Kater pendulum for precision gravity measurement
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Introduction

The gravitational acceleration (g) is a fundamental parameter which plays an important role in scientific and industrial applications. It is used, both directly and indirectly, for example to determine variation of geological rock densities, change in latitude and elevation for geophysical surveys, exploratory geological deformation, and instrumental calibration [1]. Depending on its applications, the requirement of gravity measurement accuracy could be ranging from \( 10^{-2} \) m/s\(^2\) (1 gal) to \( 10^{-6} \) m/s\(^2\) (10\(^{-4}\) gal).

There are a number of available methods to measure the gravitational acceleration. These include spring-load systems [2], time-of-flight free-fall systems [3, 4] and pendulum-based systems [5]. Although not considered a serious contender for high-precision, the pendulum-based systems have generous benefits from being straightforward to operate, low-cost maintenance and design. Additionally with advantages from computing technology, the pendulum systems can be further improved and considered a good candidate for educational and industrial purposes where cost-per-performance is a key factor.

In this study, a simple Kater pendulum is constructed as a prototype. The Kater pendulum was first made by Henry Kater. It is used to investigate and optimize the parameters to operate the pendulum in order to determine the gravitational acceleration. Several environmental parameters are also investigated and discussed in order to establish long-term repeatability of the design. Further comments are discussed as a possible future work and improvement.

Kater Pendulum

In figure 1, consider a basic physical pendulum, starting from force equation

\[
-mgd \sin \theta = I \ddot{\theta},
\]

where

- \( m \) is total mass of the pendulum
- \( g \) is the gravitational acceleration
\[ d \text{ is the distance from the center of mass to the pivot} \]
\[ I \text{ is the moment of inertia about the center of mass and equal to } mk_0^2 \text{ where} \]
\[ k_0 \text{ is the radius of gyration} \] and
\[ \theta \text{ is the angular displacement.} \]

With an assumption of small amplitude of oscillation (\( \theta \) is less than 5 degrees), one could obtain a general solution and show that the period of the pendulum is given by

\[ g = \frac{4\pi^2 I}{mdT^2}. \quad (2) \]

Equation (2) can be used to provide the value of the gravitational acceleration. However, the requirement to determine the center of mass and the exact value of moment of inertia prohibits the uselessness of this simple result. Invented by Henry Kater in 1817 [6], these problems have been circumvented by using a pendulum with two pivots at both ends, and an adjustable mass bob. By hanging the pendulum from the first pivot, the period is recorded as \( T_1 \). Then turning upside down the pendulum from the second pivot and the period is timed again as \( T_2 \). From each pivot, the period could be expressed by

\[ T_{1,2} = 2\pi \sqrt{\frac{k_0^2 + d_{1,2}^2}{gd_{1,2}}}. \quad (3) \]

where
\[ d_1 \text{ is the distance between the first pivot to the center of gravity} \]
\[ d_2 \text{ is the distance between the second pivot to the center of gravity} \]

If the mass bob’s position can be adjusted, one can readjust its position such that \( d = k_0 \) and when this condition is met, the equation (3) is simplified into \( T_{1,2} = 2\pi \sqrt{\frac{d}{g}} \), resulting in the physical pendulum behaving like a simple pendulum with physical length \( d \).
By adjusting the mass bob’s position such that the periods measured from the two pivots are equal, the distance between two pivots will be equivalent to the length of the simple pendulum and the gravitational acceleration can be determined without knowing the center of mass and the moment of inertia:

\[ T_1 = 2\pi \sqrt{\frac{k_0^2 + d_1^2}{gd_1}} \quad \text{and} \quad T_2 = 2\pi \sqrt{\frac{k_0^2 + d_2^2}{gd_2}}. \]  

(4)

If \( T_1 = T_2 \) then either \( d_1 = d_2 \) and \( d_1d_2 = k_0^2 \).

Again from equation (4),

\[ \frac{gd_1T_1^2}{4\pi^2} = k_0^2 + d_1^2 \quad \text{and} \quad \frac{gd_2T_2^2}{4\pi^2} = k_0^2 + d_2^2. \]  

(5)

Equation (5), it can be simplified into

\[ \frac{gd_1T_1^2}{4\pi^2} - \frac{gd_2T_2^2}{4\pi^2} = \left( k_0^2 + d_1^2 \right) - \left( k_0^2 + d_2^2 \right) \]  

(6)

With some algebra straightforwardly, we can arrive,

\[
g = \frac{8\pi^2}{\left( \frac{T_1^2 + T_2^2}{d_1 + d_2} \right) + \left( \frac{T_1^2 - T_2^2}{d_1 - d_2} \right)}. \]  

(7)

For the limit of \( T_1 \) approaches \( T_2 \), as \( T_1 = T_2 \), the second term of the denominator of Equation (7) will became negligibly vanished, i.e., \( \left( \frac{T_1^2 - T_2^2}{d_1 - d_2} \right) \) converges to zero subsequently, gravitational acceleration is then determined by \( g = \frac{8\pi^2}{\left( \frac{T_1^2 + T_2^2}{d_1 + d_2} \right)}. \)

Because \( d_1 \) and \( d_2 \) are distances from to the pivots to the center of mass and lie in the opposite site of the length of pendulum, the sum of \( d_1 \) and \( d_2 \) is therefore the physical length between the two pivots. Also using calculus, it can be shown that the period is minimum when \( d \) is equal to \( k_o \).

**Objective/Research Question**

The main objectives of the work are as follows,

1. To develop and design a digitization based Kater pendulum
2. demonstrate the gravity measurement with high precision for educational and industrial application
Here we raise the question; how and to what extend we can develop and optimize Kater pendulum at the accuracy and precision about $10^{-4}$

**Research Methods**

Since there are many details regarding to the methodology, we then briefly provides their information as follows. Figure (2) shows parts and components of a pendulum prototype used in this study. The pendulum is made of a rigid metal rod. The rod is threaded such that other components such as triangular-blade pivots and the mass bob can be further inserted and adjusted. The pendulum is supported by a portable stand made of rigid metal frame. In order to increase its rigidity, a deadweight of steel block in inserted at the base of the stand to reduce possible wobbling during the swing of the pendulum. In order to detect the motion of the pendulum, a pair of narrow beam laser and detector is used to detect motion of the pendulum. Once the pendulum swing through its equilibrium position, the detector sends the signal to a data acquisition card (*National Instrument*) and then to the computer. With a sampling rate of 10,000 samples per second, the time measurement error is reduced to $10^{-5}$ s. The period of the pendulum is measured over 200 cycles. This results in the accuracy of period measurement around $0.5 \times 10^{-7}$ second. The period is measure by capturing rise-up signals from the laser-based detector and using homemade software code (*LabView*) to find the average period over 200 cycles.
Figure 2: A) The Kater pendulum with rigid support table  
   B) Laser-based detector  
   C) Computer with interface and data acquisition (DAQ).
**Research procedure**

**The pendulum setup**

The pendulum is hung from one pivot and the period is measured and recorded $T_1$. Then the pendulum is turned up-side down and hung from the other pivot. The period is timed and recorded $T_2$. The mass bob is adjusted until $T_1$ and $T_2$ are roughly identical or until its difference is less than 0.01 second. The mass bob is slightly adjusted so that the graph of $T_1$ and $T_2$ as a function of the relative position of the mass bob can be plotted. To achieve high accuracy of period or the intersection of $T_1$ and $T_2$, the value of the identical period is interpolated.

In this study the following aspects are investigated:

- a) Repeatability of the measurement over a short period
- b) Repeatability of the measurement over a long period
- c) Precision of measurement

**Results**

Graph 1 and Table 1 show the result of period $T_1$ and $T_2$ as a function of the relative mass bob position. (Otherwise stated, the mass bob position is measured from the first pivot which is fixed on the metal rod.). During the investigation, the uncertainty of the period measurement is approximately around the fourth decimal place. This gives rise
to uncertainty of the gravitational measurement to around 0.000001%. The exact matching period is found by using a linear interpolation of data obtained from Table (1). It is found that the identical period is 1.3921 second. As it is no data available for the exactly value of gravitational acceleration at the location where the experiment was conducted, It is presumed that the value should lie between 9.78297 (Bangkok, Phyathai, 13° 46’ 100° 32’”) and 9.78312 (Pathumthani, 14° 02’ 100° 43’”) and the value of 9.780 are used as a nominal value.

To investigate the repeatability of the prototype, the experiment is conducted consecutively for a number of times. The standard deviation obtained from 10 trials is used an indicator against its uncertainty of measurement. For its performance against long-term variation, the experiment is repeated a few times a days over a period of one week and the standard deviation obtained from this series is used as an indicator of long-term repeatability.

**Table 1**: The results of experiment

<table>
<thead>
<tr>
<th>Position of mass from fixed pivot (in m)</th>
<th>The average of period (in second)</th>
<th>g (in m/s²)</th>
<th>The deviation from 9.780 m/s² (in %)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T1</td>
<td>T2</td>
<td></td>
</tr>
<tr>
<td>0.08700</td>
<td>1.4123</td>
<td>1.4071</td>
<td>9.5356</td>
</tr>
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<td>0.10200</td>
<td>1.4030</td>
<td>1.4019</td>
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<td>0.21750</td>
<td>1.3816</td>
<td>1.3854</td>
<td>9.9002</td>
</tr>
</tbody>
</table>

**Graph 1**: The Relation of the relative mass bob position and the period of both knife-edges.
Discussion and Conclusion
In this experiment, a gravitation measurement using a Kater pendulum has been demonstrated successfully. A Kater pendulum has been constructed as a pilot design to determine a precise value of gravitational acceleration in Thailand. From the results, by measuring and interpolating the matching periods, it helps to find the gravitational acceleration equal to 9.7780 m/s² which is around 0.0021% accuracy from the nominal reference value (9.780 m/s²). By investigating the long-term and short-term repeatability, it is found that the
uncertainty is approximately 0.011% and 0.027% respectively. It leads to a conclusion that the accuracy of the prototype can measure the gravitational constant within an accuracy of 0.0021 m/s\(^2\). Within this range of accuracy, the design has been optimized and considered useful for education and industrial applications where low-cost setup, maintenance-free portability and friendliness of operation are considered a primary trade-off. This device could be used to help establish gravity mapping in Thailand where accuracy of \(10^{-4} \text{ m/s}^2\) is required. It is believed that the prototype can be calibrated with a commercial device and then re-used as a secondary calibrator for other part of Thailand where data record of gravitational acceleration are not yet established to help mapping the gravitational field with a high spatial resolution.

**Suggestions and future work**

As there are only available data of gravity field in a few reference places in Thailand, the future work of this study will require testing the accuracy of the prototype by moving the prototype to the locations (Bangkok and Pathumthani) where record data of gravitational acceleration are available. Although the ambient temperature could results in thermal expansion by approximately \(10^{-3}\) % per Celsius, the prototype is kept in an air-conditioned environment and the thermal expansion effect is presumed negligible, the next design may include a mechanism to reduce the effect of thermal expansion.

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**References**


[14] GEOPHYSICS, VOL. 70, NO. 6 (NOVEMBER-DECEMBER 2005); P. 63ND–89ND, 6 FIGS. 10.1190/1.2133785


